Chapter 53
Integration by parts
Learning Materials

Introduction

Problems Solving
Introduction

From the product rule of differentiation:

\[ \frac{d}{dx} (uv) = v \frac{du}{dx} + u \frac{dv}{dx} \]

Where \( u \) and \( v \) are both functions of \( x \)

Rearranging gives

\[ u \frac{dv}{dx} = \frac{d}{dx} (uv) - v \frac{du}{dx} \]

Integrating both sides with respect to \( x \) gives

\[ \int u \frac{dv}{dx} \, dx = \int \frac{d}{dx} (uv) \, dx - \int v \frac{du}{dx} \, dx \]

This is known as the integration by parts formula and provides a method of integrating such products of simple functions as \( \int xe^x \, dx \), \( \int t \sin t \, dt \), \( \int e^\theta \cos \theta \, d\theta \) and \( \int x \ln x \, dx \).
Example:

Problem 1. Determine: \[ \int x \cos x \, dx \]

From the integration by parts formula,
\[ \int u \, dv = uv - \int v \, du \]

Let \( u = x \), from which \( \frac{du}{dx} = 1 \), i.e. \( du = dx \) and let
\( dv = \cos x \, dx \), from which \( v = \int \cos x \, dx = \sin x \).

Expressions for \( u, \, du \) and \( v \) are now substituted into the ‘by parts’ formula as shown below.

\[
\int \begin{bmatrix} u \\ x \end{bmatrix} \begin{bmatrix} dv \\ \cos x \, dx \end{bmatrix} = \begin{bmatrix} u \\ x \end{bmatrix} \begin{bmatrix} v \\ \sin x \end{bmatrix} - \int \begin{bmatrix} v \\ \sin x \end{bmatrix} \begin{bmatrix} du \\ (dx) \end{bmatrix}
\]

i.e. \[ \int x \cos x \, dx = x \sin x - (\cos x) + c \]
\[ = x \sin x + \cos x + c \]

[This result may be checked by differentiating the right hand side,]

i.e. \[ \frac{d}{dx}(x \sin x + \cos x + c) \]
\[ = [(x)(\cos x) + (\sin x)(1)] - \sin x + 0 \]

using the product rule

\[ = x \cos x, \text{ which is the function being integrated} \]
Example:

**Problem 5.** Determine: \( \int x^2 \sin x \, dx \)

Let \( u = x^2 \), from which, \( \frac{du}{dx} = 2x \), i.e. \( du = 2x \, dx \), and let \( dv = \sin x \, dx \), from which, \( v = \int \sin x \, dx = -\cos x \).

Substituting into \( \int u \, dv = uv - \int v \, du \) gives:

\[
\int x^2 \sin x \, dx = (x^2)(-\cos x) - \int (-\cos x)(2x \, dx)
\]

\[
= -x^2 \cos x + 2 \left[ \int x \cos x \, dx \right]
\]

The integral, \( \int x \cos x \, dx \), is not a ‘standard integral’ and it can only be determined by using the integration by parts formula again.

From Problem 1, \( \int x \cos x \, dx = x \sin x + \cos x \)

Hence \( \int x^2 \sin x \, dx \)

\[
= -x^2 \cos x + 2[x \sin x + \cos x] + c
\]

\[
= -x^2 \cos x + 2x \sin x + 2 \cos x + c
\]

\[
= (2 - x^2)\cos x + 2x \sin x + c
\]

In general, if the algebraic term of a product is of power \( n \), then the integration by parts formula is applied \( n \) times.
**Example:**

**Problem 4.** Evaluate: \( \int_0^1 5xe^{4x} \, dx \), correct to 3 significant figures

Let \( u = 5x \), from which \( \frac{du}{dx} = 5 \), i.e. \( du = 5 \, dx \) and let \( dv = e^{4x} \, dx \), from which, \( v = \int e^{4x} \, dx = \frac{1}{4}e^{4x} \)

Substituting into \( \int u \, dv = uv - \int v \, du \) gives:

\[
\int 5xe^{4x} \, dx = (5x) \left( \frac{e^{4x}}{4} \right) - \int \left( \frac{e^{4x}}{4} \right) (5 \, dx)
\]

\[
= \frac{5}{4}xe^{4x} - \frac{5}{4} \int e^{4x} \, dx
\]

\[
= \frac{5}{4}xe^{4x} - \frac{5}{4} \left( \frac{e^{4x}}{4} \right) + c
\]

\[
= \frac{5}{4}e^{4x} \left( x - \frac{1}{4} \right) + c
\]

Hence:

\[
\int_0^1 5xe^{4x} \, dx = \left[ \frac{5}{4}e^{4x} \left( x - \frac{1}{4} \right) \right]_0^1
\]

\[
= \left[ \frac{5}{4}e^{4} \left( 1 - \frac{1}{4} \right) \right] - \left[ \frac{5}{4}e^{0} \left( 0 - \frac{1}{4} \right) \right]
\]

\[
= \left( \frac{15}{16}e^{4} \right) - \left( -\frac{5}{16} \right)
\]

\[
= 51.186 + 0.313 = 51.499 = 51.5,
\]

correct to 3 significant figures.
Example:

**Problem 7.** Determine: $\int \ln x \, dx$

$\int \ln x \, dx$ is the same as $\int (1) \ln x \, dx$

Let $u = \ln x$, from which, $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$ and let $dv = 1 \, dx$, from which, $v = \int 1 \, dx = x$

Substituting into $\int u \, dv = uv - \int v \, du$ gives:

$$\int \ln x \, dx = (\ln x)(x) - \int x \, \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + c$$

Hence $\int \ln x \, dx = x(\ln x - 1) + c$

**Problem 6.** Find: $\int x \ln x \, dx$

The logarithmic function is chosen as the ‘$u$ part’ Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$

Letting $dv = x \, dx$ gives $v = \int x \, dx = \frac{x^2}{2}$

Substituting into $\int u \, dv = uv - \int v \, du$ gives:

$$\int x \ln x \, dx = (\ln x) \left( \frac{x^2}{2} \right) - \int \left( \frac{x^2}{2} \right) \, \frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \left( \frac{x^2}{2} \right) + C$$

Hence $\int x \ln x \, dx = \frac{x^2}{2} \left( \ln x - \frac{1}{2} \right) + c$

or $\frac{x^2}{4} (2 \ln x - 1) + c$
Exercise:

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x \, dx$ [16.78]
7. $\int_0^{\frac{\pi}{4}} x \sin 2x \, dx$ [0.2500]
8. $\int_0^{\frac{\pi}{2}} t^2 \cos t \, dt$ [0.4674]
9. $\int_1^2 3x^2e^\frac{x}{2} \, dx$ [15.78]

Exercise 186  Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int xe^{2x} \, dx = \left[ \frac{e^{2x}}{2} \left( x - \frac{1}{2} \right) + c \right]
2. \int \frac{4x}{e^{3x}} \, dx = \left[ -\frac{4}{3}e^{-3x} \left( x + \frac{1}{3} \right) + c \right]
3. $\int x \sin x \, dx = [-x \cos x + \sin x + c]
4. $\int 5\theta \cos 2\theta \, d\theta = \left[ \frac{5}{2} \left( \theta \sin 2\theta + \frac{1}{2} \cos 2\theta \right) + c \right]
5. $\int 3t^2e^{2t} \, dt = \left[ \frac{3}{2}e^{2t} \left( t^2 - t + \frac{1}{2} \right) + c \right]$
Exercise:

Exercise 187  Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. \[ \int 2x^2 \ln x \, dx \left[ \frac{2}{3} x^3 \left( \ln x - \frac{1}{3} \right) + c \right] \]

2. \[ \int 2 \ln 3x \, dx \left[ 2x(\ln 3x - 1) + c \right] \]

3. \[ \int x^2 \sin 3x \, dx \left[ \frac{\cos 3x}{27} (2 - 9x^2) + \frac{2}{9} x \sin 3x + c \right] \]

4. \[ \int 2e^{5x} \cos 2x \, dx \left[ \frac{2}{29} e^{5x} (2 \sin 2x + 5 \cos 2x) + c \right] \]

5. \[ \int 2\theta \sec^2 \theta \, d\theta \left[ 2[\theta \tan \theta - \ln(\sec \theta)] + c \right] \]

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. \[ \int_1^2 x \ln x \, dx \quad [0.6363] \]

7. \[ \int_0^1 2e^{3x} \sin 2x \, dx \quad [11.31] \]

8. \[ \int_0^\frac{\pi}{2} e^t \cos 3t \, dt \quad [-1.543] \]

9. \[ \int_1^2 x^2 \ln x \, dx \quad [12.78] \]

10. In determining a Fourier series to represent \( f(x) = x \) in the range \(-\pi\) to \(\pi\), Fourier coefficients are given by:

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx \]

and
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx
\]

where \( n \) is a positive integer. Show by using integration by parts that \( a_n = 0 \) and \( b_n = -\frac{2}{n} \cos n\pi \)

11. The equations:

\[
C = \int_0^1 e^{-0.4\theta} \cos 1.2\theta \, d\theta
\]

and
\[
S = \int_0^1 e^{-0.4\theta} \sin 1.2\theta \, d\theta
\]

are involved in the study of damped oscillations. Determine the values of \( C \) and \( S \).

\( [C = 0.66, S = 0.41] \)